FOURIER DESCRIPTORS OF TWO DIMENSIONAL SHAPES -RECONSTRUCTION AND ACCURACY *

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ABSTRACT

Two kinds of Fourier shape descriptors (FD's) are considered: ZR defined by Zahn and Roskies and G defined by Granlund. In the first part of the paper ZR descriptors are studied. Three modifications of ZR descriptors are proposed. The new descriptors are based on the smoothed signature, linearized smoothed signature and the curvature function. The amplitudes of Fourier descriptors are shown to be invariant under rotations, translations, changes in size, mirror reflections and shifts in the starting point. In all cases the reconstruction accuracy in terms of the number of Fourier descriptors is studied resulting in approximation error bounds. An efficient reconstruction method not requiring numerical integration is proposed for polygonal shapes. It also provides polygonal approximation for arbitrary contours. In the second part of the paper theoretical results are verified in numerical experiments involving digitized patterns.

INTRODUCTION

In many applications of pattern recognition and digital image analysis the shape of a simply connected object is represented by its contour. Different approaches have been proposed for 2-D contour analysis, they include statistical approaches based on the method of moments, Fourier descriptors [1, 5, 6, 8, 11], curve signatures [7], circular autoregressive models [2], syntactic approaches [3] and relaxation approaches. In the statistical approach including Fourier descriptors, moments and autoregressive models, numerical features are computed from the complete boundary and statistical discriminators are used to classify contours. Among different techniques, Fourier descriptors (or simple quantities derived from them) are distinguished by their invariance to affine shape transformations (scaling, rotation, translation and mirror reflections) and to shifts in the starting point [5, 8, 11]. In the literature the popularity of G descriptors [4, 5, 8] far exceeds that of ZR descriptors [9, 11]. One of the reasons is the discontinuity of the polygonal signature resulting from the ZR approach, which causes the Fourier coefficients to decrease slowly. Another reason is that time-consuming numerical integration is used for the reconstruction in the ZR method.

In this paper the problems mentioned above are resolved providing answers to the open questions posed in [8, 11]. The results of this paper show that the methods using ZR Fourier descriptors as well as the ones proposed in this study give a better reconstruction rate than the G method for the small number of FD's. For the ZR approach the bounds on the deviation between a contour and its approximation based on the finite number of Fourier coefficients are derived. These bounds tell us how many FD's are needed to achieve a given accuracy in the reconstruction, answering the question posed in [11]. Surprisingly, the rate of reconstruction for the ZR approach is asymptotically the same as the one obtained for the G method [1, 4]. Next, three improvements of the ZR approach for polygonal contours are proposed. The angular bend signature used in [11] has jump discontinuities for polygonal curves. It is well known [10] that the partial Fourier series derived from the discontinuous function does not converge uniformly to its limit at the jump points (Gibbs phenomenon). First we replace the linearized signature used in [11] by the smoothed signature (SZR) with controlled degree of smoothing and linearized smoothed signature (LSZR). The first signature is obtained by smoothing the angular bend function the other by linearizing SZR. Smoothing of the signature results in rounded edges near polygonal vertices. Since LSZR is continuous the reconstruction bounds known in the literature [1, 4] become applicable to it. Finally, we propose a contour signature (CS) based on the curvature function. Since for polygonal curves the curvature is zero along the sides and infinite at the vertices we replace the curvature function by its smooth approximation using a finite Fourier series. We believe that CS descriptors may be very useful in shape classification since they are sensitive to sharp changes in the contours. For all three signatures we derive Fourier descriptors and reconstruction rates. We also propose reconstruction formulae that do not require numerical integration which results in considerable savings in computer time. The ZR and LSZR methods are tested and compared with G approach in reconstruction experiments with digitized handwritten characters. For the purpose of comparison several similarity measures are used. The obtained results can be applied in shape recognition and classification.

ZR DESCRIPTORS-EFFICIENT RECONSTRUCTION AND BOUNDS

Zahn and Roskies [11] defined Fourier descriptors as follows. Let γ be a clockwise-oriented simple, closed,smooth curve of length L with parametric representation $Z(\ell)=(x(\ell), y(\ell))$ where ℓ is an arc length and $0 \le \ell \le L$. Also, let $\theta(\ell)$ be the angular direction of γ at point ℓ. The cumulative angular bend function $\phi(\ell)$ is defined as the net amount of angular bend between the starting point $\ell=0$ and point ℓ. So $\phi(\ell)=\theta(\ell)-\theta(0)$ except for possible multiples of 2π and $\phi(L) = -2\pi$. In [11] a curve signature was defined as

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$$\phi^*(t) = \phi(\frac{Lt}{2\pi}) + t, \ 0 \leq t \leq 2\pi. \qquad (1)$$

Clearly, $\phi^*(t)$ is invariant under rotation, translation and scaling, making it a good candidate for a shape signature. Expanding ϕ^* as a Fourier series in amplitude–phase form we have

$$\phi^*(t) = \mu_0 + \sum_{k=1}^{\infty} A_k \cos(kt-\alpha_k) \qquad (2)$$

and $\{A_k, \alpha_k\}_1^{\infty}$ is the set of Fourier descriptors (ZR descriptors) of the curve γ. Let ϕ_n^* denote the Fourier series in equation (2) truncated to the first n terms. For the signature in equation (1) reconstruction formula using an integral was suggested in [11, (eq.) 5]. The use of an integral in the reconstruction formula is one of the disadvantages of the ZR approach due to the long computation time required [8]. We propose a simplified reconstruction formula for an important class of polygons. Let γ be a polygon with m vertices $V_0, V_1, \cdots, V_{m-1}, V_m=V_0$ and edges (V_{i-1}, V_i) of length $\Delta\ell_i$, $i=1,\cdots,m$ $(\Delta\ell_0=0)$. The angular change of direction at a vertex V_i is $\Delta\phi_i$ and $L = \sum_{i=1}^{m} \Delta\ell_i$. Define $\ell_i = \sum_{j=1}^{i} \Delta\ell_j$, $\ell_0=0$, $\Omega_i = \frac{2\pi}{L}[\ell_{i-1}, \ell_i]$, $c_i = \sum_{j=1}^{i-1} \Delta\phi_j$. and assume also that $\phi(0) = \Delta\phi_0/2$. We note that $\phi^*(t) = \sum_{i=1}^{m} (t+c_i)\ I_{\Omega_i}(t)$, where $I_\Omega(\cdot)$ stands for the characteristic function of the set Ω.

Let $\ell \in [\ell_s, \ell_{s+1}]$, s= 0,$\cdots$, m-1. We propose a **fixed increment** approximate reconstruction formula which can be applied to arbitrary and not necessary polygonal curves.

$$Z(\ell) = Z(0) + \Delta\ell \sum_{j=1}^{s} \exp\{\ i(\delta_0 + \hat{c}_j)\ \}$$
$$+ (\ell-\ell_s) \exp\{\ i(\delta_0 + \hat{c}_{s+1})\ \} \qquad (3)$$

where $\Delta\ell$ is an arbitrary fixed length, $\ell_s = \Delta\ell\ \mathrm{Int}[\ell/\Delta\ell]$,

$$\hat{c}_j = \phi^*_n(m_j) - m_j,$$

and

$$m_j = \frac{2\pi}{L}\ ((j-1)\Delta\ell + \frac{\Delta\ell}{2}) = \frac{2\pi\Delta\ell}{L}\ (j-\frac{1}{2}).$$

Formula (3) works as follows. Starting at Z(0) we move by constant steps of the length $\Delta\ell$ in the directions $\hat{c}_j$ indicated by the angular bend function in jth interval until $0<\ell-\ell_s<\Delta\ell$ when the step size becomes $\ell-\ell_s$ and the direction is $\hat{c}_{s+1}$. The curve reconstructed by eq. (3) is a polygon. For arbitrary curve (3) gives a polygonal approximation. Formula (3) is approximate even for polygons with coefficients c_j known, unless the length of every side of the polygon is an integer multiple of $\Delta\ell$. The simulation results in the last section show excellent reconstruction performance of formula (3) with the time complexity substantially lower than for formula (5) in [11]. The signature ϕ^* derived from a polygonal curve contains jump discontinuities at the points corresponding to the vertices of the polygon. As pointed out in [11] these discontinuities cause the Fourier coefficients in equation (1) to decrease more slowly than the corresponding coefficients in the G method. However, surprisingly, we found that γ can be reconstructed using ZR descriptors at the same asymptotical rate as with the G descriptors, except in the neighbourhood of the finite number of points of discontinuity at which convergence does not take place. Let Z_n denote the curve corresponding to the finite set of ZR descriptors $\{A_k, \alpha_k\}_1^n$.

$$||Z_n - Z||_\Omega = \sup_{\ell\in\Omega} |Z_n(\ell)-Z(\ell)| \leq L\ \frac{c}{n} \qquad (4)$$

where

$$\Omega = \bigcup_{i=1}^{m} \Omega_i^{\delta}, \qquad \Omega_i^{\delta} = [\ell_{i-1}+\delta,\ \ell_i-\delta],\ \delta > 0$$

and c is a constant depending upon ϕ^* and δ.
All points of discontinuity of ϕ^* are excluded from Ω because the Fourier series of a discontinuous function does not converge uniformly to the function due to the Gibbs phenomenon at the jump points [10]. The theorem implies that the rate of reconstruction in the ZR method is comparable to that of the G method for large n. For a small number of FD's the ZR method gives better accuracy of the reconstruction. This claim has been confirmed by numerical experiments described in the last section.

MODIFICATIONS AND IMPROVEMENTS OF ZR DESCRIPTORS

Smoothed signature

Let γ be a polygonal curve. Both ZR and STZR descriptors discussed so far are derived from discontinuous signatures resulting in the slower decline of the Fourier coefficients and nonconvergence of the partial Fourier series at the jump points. To remedy this situation we introduce a smoothed signature (SZR).

Definition. Consider a closed, polygonal curve γ with vertices $V_0,\ldots,V_m=V_0$. Smoothing of γ of order δ is obtained by inserting circular arcs $T_i = A_iV'_iB_i$ $i=0,\ldots,m-1$ of length $\Delta\ell'_i = \delta\Delta\phi_i \mathrm{ctg}\frac{\Delta\phi_i}{2}$ into angular corners so that the resulting quasipolygonal curve γ' is differentiable everywhere. We arbitrarily assume that arc centers V'_i correspond to polygonal vertices V_i, $i=0,\ldots,m-1$.

Let ℓ'_i be the length of the smoothed polygon between V'_0 and V'_i. The lengths of inserted arcs T_i are smaller than 2δ–the lengths of corresponding angular corners $A_iV_iB_i$. Consequently we obtain a closed contour γ' of length $L' = L-2m\delta+\sum_{i=1}^{m} \Delta\ell'_i$, which is less than L–the length of γ. SZR is a continuous function on the interval $[0, L')$, where L' is the length of the quasipolygonal curve. We can calculate the Fourier descriptors for the polygonal contours by expanding $\bar{\phi}(\ell)$ as a Fourier series in the amplitude–phase angle form

$$\bar{A}_k = (\bar{a}_k^2 + \bar{b}_k^2)^{\frac{1}{2}}, \quad \bar{\alpha}_k = \text{arctg}\,\frac{\bar{b}_k}{\bar{a}_k}, \quad k=1, 2, \cdots$$

provided that $\bar{a}_k \neq 0$. Coefficients $\bar{\mu}_0$, $\bar{a}_k$, $\bar{b}_k$ are defined below.

$$\bar{\mu}_0 = -\frac{1}{L'}\sum_{i=1}^{m}\ell_i'\Delta\phi_i - 2\pi + \frac{\Delta\phi_m'}{2}$$

$$\bar{a}_n = \frac{1}{\pi n}\left[-\frac{L'}{\pi n}\sum_{i=1}^{m-1}\frac{\Delta\phi_i}{\Delta\ell_i'}\sin\frac{2\pi n\ell_i'}{L'}\sin\frac{\pi n\Delta\ell_i'}{L'}\right]$$

$$\bar{b}_n = \frac{1}{\pi n}\left[\frac{L'}{\pi n}\sum_{i=1}^{m}\frac{\Delta\phi_i}{\Delta\ell_i'}\cos\frac{2\pi n\ell_i'}{L'}\sin\frac{\pi n\Delta\ell_i'}{L'} + 2\pi\right]$$

SZR has one disadvantage: its periodic extension on the whole real line has jump discontinuities at multiplicities of L' since $\bar{\phi}(0) = 0 \neq \bar{\phi}(L') = -2\pi$. To remove these discontinuities we define a linearized, smoothed signature (LSZR) normalized in the interval $[0, 2\pi]$ as follows

$$\psi^*(t) = \bar{\phi}\left[\frac{L't}{2\pi}\right] + t.$$

Signature ψ^* is invariant under translations, rotations and changes of scale. We expand ψ^* in a Fourier series and after straightforward calculations we establish a simple relationship between the Fourier coefficients of $\bar{\phi}$ and ψ^*

$$\bar{\mu}^* = \bar{\mu} + \pi, \; \bar{a}_n^* = \bar{a}_n, \; \bar{b}_n^* = \bar{b}_n - 2/n.$$

Obviously, $\bar{a}_n^*$ and $\bar{b}_n^*$ are functions of δ and they converge to ZR Fourier coefficients as $\delta \to 0$. FD's corresponding to LSZR are defined as follows

$$\bar{A}_n^* = (\bar{a}_n^{*2} + \bar{b}_n^{*2})^{\frac{1}{2}}, \quad \bar{\alpha}^* = \text{arctg}\,\frac{\bar{b}_n^*}{\bar{a}_n^*}, \quad n=1, 2, \cdots$$

given that $\bar{a}_n^* \neq 0$. In order to apply LSZR effectively we can use efficient reconstruction formula (3) with $\hat{c}_j = \psi^*_n(m_j) - m_j$. Since ψ^* is continuous on the whole real line its FD's converge faster than the ones derived from the previously discussed signatures. That implies a better reconstruction rate. To obtain the reconstruction bounds we can apply the bounds from [1, 4] which are valid for continuous functions. Thus we have

$$\|Z_n - Z\|_{[0,\,L']} \leq \frac{L'}{\pi}\,\frac{\text{Var}\;\dot{\psi}^*}{n}$$

where Var denotes the total variation, $\dot{\psi}^*$ stands for the derivative of ψ^* and Z_n is the curve reconstructed from $\{\bar{A}_k^*, \bar{\alpha}_k^*\}_1^n$. It is easy to show that Var $\dot{\psi}^* = \sum_{i=1}^{m}\frac{|\Delta\phi_i|}{\Delta\ell_i'}$ (i.e. the sum of slopes of linear pieces in $\bar{\phi}$). LSZR descriptors behave similarly under affine transformations as SZR descriptors. The results are summarized below.

<u>Theorem</u>. If γ and γ' are two curves which differ by $\Delta\ell$ in the starting points Z_0 and Z_0' then $\{A_n^*, \alpha_n^*\}$ and $\{A_n^{*\prime}, \alpha_n^{*\prime}\}$ for γ and γ' satisfy:

i) $A_n^{*\prime} = A_n^*$, ii) $\alpha_n^{*\prime} = \alpha_n^* - n\Delta t$

iii) $\mu_0^{*\prime} = \mu_0^* + \delta_0^* - \delta_0^{*\prime} - \Delta t$

where $\Delta t = \frac{2\pi\Delta\ell}{L'}$.

If γ and γ' are mirror reflections of one another then

i) $A_n^{*\prime} = A_n^*$, ii) $\alpha_n^{*\prime} = -\alpha_n^* - \pi$

iii) $\mu_0^{*\prime} = -\mu_0^* - \Delta\phi_0$.

<u>Signature based on the curvature</u>

In earlier sections we considered signatures based on the angle versus length function $\theta(\ell)$. Since essential shape information is contained in the curvature of the contour we will analyse FD's based on the curvature signature (CS) defined as $k(\ell) = \frac{d\theta(\ell)}{d\ell} = \frac{d\phi(\ell)}{d\ell}$. The problem with CS lies in that $k(\ell)$ is equal to zero almost everywhere for polygonal curves and therefore CS is not a good candidate for deriving Fourier descriptors. We can avoid that problem by defining CS based on the Fourier approximation of the normalized STZR, that is by using $\phi_n^*(t) = \hat{\mu}_0 + \sum_{k=1}^{n}\hat{A}_k\cos(kt - \hat{\alpha}_k)$, where $\{\hat{A}, \hat{\alpha}_k\}$ are descriptors defined in (9) and $t = 2\pi\ell/L$. In order to obtain size-invariant curvature descriptors we define the normalized curvature function $k^*(t) = Lk(Lt/2\pi)$. Using $\phi_n^*(t)$ we obtain

$$\hat{k}^*(t) = \frac{d\phi_n^*(t)}{dt} = \sum_{k=1}^{n}\hat{B}_k\cos(kt - \hat{\beta}_k)$$

where $\hat{B}_k = 2\pi k\hat{A}_k$, $\hat{\beta}_k = \hat{\alpha}_k - \pi/2$.

As a result we obtain FD's $\{\hat{B}_k, \hat{\beta}_k\}_1^n$ based on the curvature function. The main difference between CS descriptors and SZR descriptors is the greater weight which amplitudes of CS descriptors give to higher frequencies. That may result in the faster reconstruction of the details on the contour. In order to reconstruct the curve from CS descriptors we can use the formula introduced in [4, eq. (4)]

$$Z(\ell) = Z_0 + \int_0^{\ell}\exp\{i\,(\int_0^{\lambda}k(t)dt + \delta_0)\}\,d\lambda.$$

The reconstruction of polygonal curves represented by CS descriptors may be achieved by using formula (3) with c_j estimated by $\hat{c}_j = \frac{L}{2\pi j}\hat{k}^*(m_j)$, where $m_j = \frac{2\pi}{L}(\ell_{j-1}+\Delta\ell_j/2)$ or $m_j = \frac{2\pi\delta\ell}{L}(j-1/2)$. Since CS descriptors are derived from the approximate cumulative bend function ϕ bound (4) is valid for CS descriptors with constant c depending upon ϕ and δ. The harmonic amplitudes of CS FD's show the same invariance for affine transformations as descriptors discussed in the previous sections.

RECONSTRUCTION OF HANDWRITTEN CHARACTERS

In order to verify the results of the preceding sections we performed a number of experiments on a set of digitized handwritten characters selected from Suen's data base [6]. Experiments that we performed indicate that the ZR method is more efficient than the G method in the reconstruction of curves. For comparison the reconstructions of letter B and S using LSZR descriptors are shown in Figure 1. The usefulness of the fixed increment reconstruction formula (eq. (3)) using ZR descriptors is demonstrated for characters B and S in Fig. 2. The results confirm that for the appropriate choice of the step size $\Delta\ell$, the reconstruction performances of formulae (5) in [11] and (3) are identical while the latter is more efficient.

In order to apply the FD's in pattern recognition we have to decide how many descriptors carry sufficient shape information and then use these FD's as coordinates of the feature vector. In order to compare different methods we introduced the following similarity measures.

1) Length of the reconstructed curve. The length of original curve was used in the reconstruction formula.

2) Mean integrated absolute deviation between the reconstructed curve and original curve.

3) Mean integrated squared deviation between the reconstructed approximation and the original.

In all instances LSZR descriptors showed the best performance. For a small number of FD's (i.e. less than 10) ZR and LSZR methods outperform the G method for all three measures. That may suggest the usefulness of LSZR descriptors in shape representation and recognition when it is important to use the small sizes of feature vectors. For a moderate number of descriptors the G approach gives only slightly worse performance than LSZR approach. The ZR method remains clearly behind the other two. We notice that the sizes of reconstructed shapes are magnified by ZR type descriptors but are reduced by G descriptors. The simulation was performed on the MicroVax II workstation.

CONCLUSIONS

In this paper Granlund and Zahn–Roskies descriptors were studied. Several modifications of ZR descriptors were proposed leading to continuous signatures and better reconstruction rates. New, efficient reconstruction formulae were also proposed. LSZR descriptors show the best performance among the descriptors studied in this paper. The performance of shape descriptors was measured by several similarity measures and was evaluated in experiments involving handwritten samples. Good performance of LSZR descriptors makes them good candidates for shape features. The applications in shape classification will be investigated.

REFERENCES

[1] F. M. Dekking and P. J. Van Otterloo, "Fourier coding and reconstruction of complicated contours", IEEE Trans. on Systems, Man, and Cybernetics, Vol. SMC-16, No. 3, May/June 1986.

[2] S. R. Dubois and F. H. Glanz, "An autoregressive model approach to 2-D shape classification", IEEE Trans. Pattern Anal. Machine Intel., 1986, pp. 55-66.

[3] K. S. Fu, "Syntactic Pattern Recognition", Prentice Hall, Englewood Cliffs, 1982

[4] C.R. Giardina and F. P. Kuhl, "Accuracy of curve approximation by vectors with elliptical loci", Comp. Graph. Image Proc. Vol. 6, 1977, pp. 277-285.

[5] G. H. Granlund, "Fourier preprocessing for hand print character recognition", IEEE Trans. Comp., Vol. C-21, 1972, pp. 195-201.

[6] M.T.Y. Lai and C. Y. Suen, "Automatic recognition of characters by Fourier descriptors and boundary line encodings", Pattern Recognition Vol. 14, No's. 1-6, 1981, pp. 383-393.

[7] J. O'Rourke, "The signature of a plane curve", SIAM J. Comput., Vol. 15, No. 1, February 1986.

[8] E. Persoon and K. S. Fu, "Shape discrimination using Fourier descriptors:, IEEE Trans. Syst. Man, Cybern. Vol. SMC-7, March 1977, pp. 170-179.

[9] J. Strackee and N. J. D. Nagelkerke, "On closing the Fourier descriptor presentation", IEEE Trans. on Pattern Analysis and Machine Intelligence, Vol. PAMI-5, No. 6, November 1983.

[10] G. P. Tolstov, Fourier Series, Dover, New York, 1976.

[11] C. T. Zahn and R. S. Roskies, "Fourier descriptors for plane closed curves", IEEE Trans. Comput., Vol. C-21, March 1972, pp. 269-281.

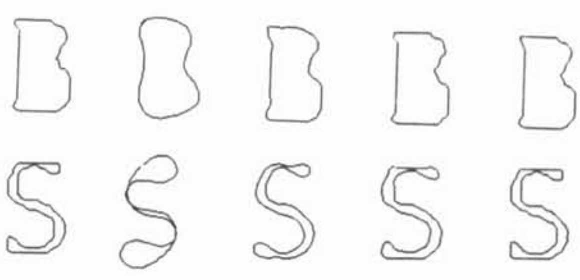

Figure 1. Reconstructed characters B and S using the LSZR method and eq. (5) in [11] with the number of FD's equal to 5, 13, 25 and 65 respectively. Smoothing parameter δ=0.5.

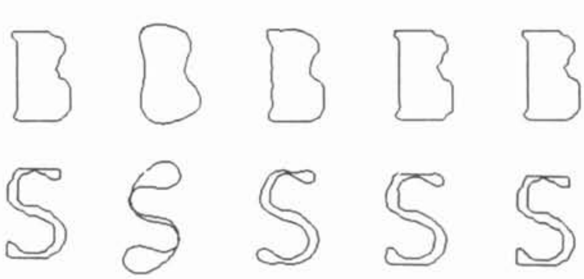

Figure 2. Reconstructions of contours B and S using LSZR descriptors with the similar number of FD's as in Fig. 1 and eq. (3) with $\Delta\ell$=0.53 and δ=0.5