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Measuring the Complexity of Two-Dimensional Binary Patterns —Sub-Symmetries versus Papentin Complexity—

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Abstract

This paper describes an experimental comparison of two measures of the complexity of binary patterns with respect to how well they predict human judgement of visual complexity. The experiments are performed with a data set consisting of 45 binary patterns defined on a square 6×6 array of black and white squares. The measures compared are generalizations of the measures previously explored for one-dimensional binary sequences by Alexander and Carey as well as Papentin. The former is based on counting the number of sub-symmetries present in the pattern, and the latter is an upper bound on the Kolmogorov complexity. This upper bound is obtained by calculating the shortest length of all possible descriptions of the pattern among a hierarchy of description languages.

1 Introduction

The complexity of a pattern is one of its most salient structural characteristic features, and it is thus no surprise that complexity measures find a wide variety of practical applications to diverse areas such as letter identification [14], handwriting recognition [15], information retrieval [17], data compression [9], computational music [18], and psychology [2], [4]. An extensive review of measures of visual complexity is found in reference [5]. One property of patterns that forms an integral part of several measures of pattern complexity is mirror symmetry [8]. A very simple and largely forgotten measure of complexity of one-dimensional visual sequences is the measure of Alexander and Carey which counts the total number of sub-symmetries present in the sequence [2]. Patterns with mirror symmetry are generally deemed simpler than patterns that do not contain such symmetries. Another popular approach to measure the complexity of binary sequences is based on lossless compression of the sequence. The complexity of the sequence is then measured by the length of the compressed version. This approach is most notably exemplified by the Kolmogorov complexity [7], [9]. In practice however, the Kolmogorov complexity is not computable, and hence it must be approximated in some way, usually by computing an upper bound instead. An attractive and practical approach to computing an upper bound on the Kolmogorov complexity, via a hierarchy of description languages, was proposed by Papentin [12], [13].

In this paper the measures of complexity of Papentin and Alexander-Carey, originally proposed for one-dimensional binary sequences, are compared with each other and with human judgments of complexity, using a dataset of two-dimensional visual patterns. This dataset consists of 45 patterns defined on a 6×6 square array of black and white squares.

2 Pattern Sub-Symmetries

2.1 One-dimensional pattern sub-symmetries

In 1968 Christopher Alexander and Susan Carey proposed an amazingly simple empirical measure of the complexity of visual one-dimensional binary patterns that consist of a concatenation of squares colored either black or white [2]. What is more noteworthy is that in the experiments performed with human subjects they found that the measure correlated highly with human judgments of cognitive complexity (0.808, significant at the 0.00001 level). The measure is quite simple: it just counts the total number of sub-symmetries present in the pattern. A sub-symmetry is a contiguous subset of adjacent squares that possesses mirror symmetry. In the computer science theory of words, the same notion is called a *palindrome*, and the total number of palindromes contained in a sequence is called the *palindrome complexity* [1]. The example in Figure 1 comparing two sequences of seven squares each, illustrates the procedure. The sequence on the left has four sub-symmetries of length 2, and one each of lengths 3, 5, and 7, for a total of 7. The sequence on the right has one sub-symmetry of length 2, three of length 3,

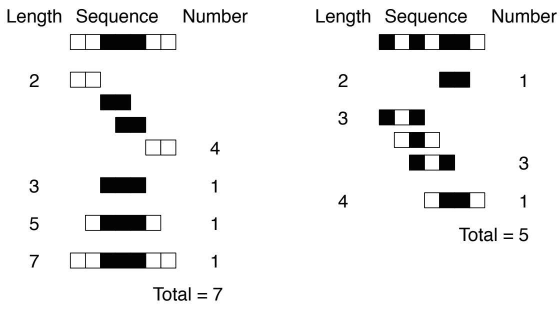

Figure 1. Calculation of sub-symmetries.

and one of length 1, for a total of 5. A pattern with a relatively greater number of sub-symmetries is considered simpler than one with fewer sub-symmetries, and thus the pattern on the right is deemed more complex than the one on the left.

The utter simplicity of this measure, its inherent natural coding of nested hierarchical symmetries contained in a pattern, and its high correlation with human judgments in the visual domain, motivated its exploration in the auditory world of musical rhythms. It was found that if musical rhythms are notated in box-notation, the measure then also correlates highly with various empirical tests of rhythm performance and cognition obtained by means of listening and reproduction experiments with human subjects [18].

2.2 Two-dimensional pattern sub-symmetries

Inspired by the work of Alexander and Carey, Susan Chipman explored several measures of complexity of two-dimensional binary patterns [4]. The measure that correlated most highly with human judgments of complexity (correlation = 0.72) generalizes the one-dimensional Alexander-Carey measure to two dimensions by applying it successively to each of the rows and columns of the pattern, with the goal of measuring the overall vertical and horizontal sub-symmetries. They modified the original measure of complexity proposed by Alexander and Carey by defining it as the weighted linear combination of all the symmetric subsequences found, where each is weighted by its length (the number of elements or squares). This weighting scheme is equivalent to counting all the squares that make up all the sub-symmetries found. For the two one-dimensional patterns of Figure 1, the weighted measures yield the values 23 and 15 for the patterns on the left and right, respectively. Presumably this weighting scheme was used based on the assumption that the presence of relatively large symmetric subsets of a pattern are more indicative of overall simplicity than relatively small subsets. On the other hand, this weighting scheme emphasizes the presence of exact global symmetries and downplays the hierarchical role of small and medium sized sub-symmetries (local symmetries). Hierarchy is a psychologically relevant notion [6], and furthermore, exact symmetries in the real world are hard to find, thus highlighting the importance of hierarchical symmetry measures that are sensitive to approximate symmetries [20]. One goal of the present study is to determine whether the weighting scheme adopted by Chipman offers any advantages over treating sub-symmetries equally at all hierarchical levels.

3 Papentin Complexity

3.1 One-dimensional Papentin complexity

A completely different approach to measuring the complexity of one-dimensional patterns is via the length of the shortest possible algorithm or computer program that will generate the pattern. This measure is referred to most frequently as the Kolmogorov complexity, and was proposed independently in different guises by Solomonoff [16], Chaitin [3], and Kolmogorov [7]. Although this idea is powerful and fruitful in theory, it is not computable, and therefore not practical [9]. Nevertheless, viable methods exist for approximating the Kolmogorov complexity via upper bounds. Frank Papentin proposed one such measure defined as the shortest length of all possible descriptions in a hierarchy of description languages [12], [13]. Let $X = (x_1, x_2, ..., x_n)$ denote a sequence of n binary-valued symbols. At the first and lowest hierarchical level, the Papentin complexity denoted by L_0 is defined as the length (number of symbols) of the sequence. Thus $L_0(X) = n$. Intuitively, as a crude first approximation, it is reasonable to suppose that a long sequence should be more complex than a short one. However, this is not necessarily so. A sequence such as *abbabaaababbbb* is much shorter than *ab* repeated one thousand times, but the latter is simpler and easier to describe than the former. At the second level in the hierarchy, the complexity, denoted by L_1, is defined as the length of the description of the sequence in terms of the number and size (length) of the runs of identical symbols in the sequence. Consider the sequence $X = aaabaabaab$. Repetitions of the same symbol are described by an exponent, so that $X = a^3ba^2ba^2b$. Furthermore, let n_r denote the number of runs in the sequence, m_i the number of times a symbol is repeated in the ith run, and k the number of runs of length greater than one. For $X = aaabaabaab$ we have that $n_r = 6$, $m_1 = 3$, $m_2 = 2$, and $m_3 = 2$. Then the complexity at this hierarchical level is given by the equation:

$$L_1(n) = n_r + \sum_{i=1}^{k} \log m_i \tag{1}$$

Substituting for the values of n_r, k, and m_i of the sequence X yields:

$$L_1(n) = 6 + \log 3 + 2\log 2 = 7.079 \tag{2}$$

At the third hierarchical level of description languages, subsequences that repeat in the sequence may be replaced by the symbol S_i, but this symbol must become part of the new description, by appending it and separating it by a comma in the description itself. As an example, consider the sequence $Y = abab bbbabababababbbba$, in which the subsequence *ababbbb* occurs twice, and the subsequence *ba* occurs three times. If we replace *ababbbb* by S_1, and *ba* by S_2, we obtain the sequence:

$$S_1S_2S_2S_2bS_1a, ababbbb, ba \tag{3}$$

which may be shortened further using the same rules used in the second hierarchy to:

$$S_1S_2^3bS_1a, abab^4, ba \tag{4}$$

which yields the value:

$$L_2(n) = 13 + \log 3 + \log 4 = 14.72 \tag{5}$$

It should be pointed out that the above definitions of S_1 and S_2 represent only two choices among the many other alternatives available, and all such possibilities may need to be examined in order to be sure that the shortest description is found.

Papentin's fourth hierarchical level $L_3(n)$ incorporates two transformations of subsequences: mirror symmetry and complementation. The sequence *aaabaabaabbaabaabaaa* contains the subsequence *aaabaabaab* followed by its inversion *baabaabaaa*. It therefore has mirror symmetry, and is therefore a palindromic subsequence. This transformation is denoted by the symbol $\Leftarrow$. In the complementation operation the symbol a is exchanged with symbol b and vice versa. For instance, the sequence *aaabaabaabbbbabbabba* contains *aaabaabaab* followed by its complement *bbbabbabba*. This transformation is denoted by the symbol $\Uparrow$. Thus, the sequence *abbbabbaababbbaabbabbbabaaabaabba* may be written as:

$$Sab \Leftarrow S \Uparrow S, ab^3ab^2a^2b \tag{6}$$

which yields the value:

$$L_3(n) = 14 + \log 3 + 2\log 2 = 15.079 \tag{7}$$

The process of coding more and more properties of order and types of repetition to obtain a shorter description has no end, making the Kolmogorov complexity not computable. However Papentin's procedure is one way of obtaining an upper bound on the Kolmogorov complexity, and thus gives a computable and practical approximation. The more time and ingenuity one expends the sharper the bound obtained. For further hierarchical levels of description languages, the reader is referred to the papers by Papentin [12], [13]. However, it was found through experimentation that for the short sequences that make up in the datasets used in the present study, the shortest descriptions were obtained with the L_1 measure. Furthermore, this measure has the added attractive feature that it is computationally very efficient. A simple traversal of the sequence yields the number of runs of identical symbols, as well as the number of repetitions in each run, and therefore $L_1(n)$ defined in Equation 1 may be computed in $O(n)$ worst-case time. By contrast, the straightforward approach to calculating the weighted sum of all the sub-symmetries takes $O(n^2)$ time, and even the most efficient $O(n)$ algorithm of Manacher for just counting the sub-symmtries is complicated [10].

3.2 Two-dimensional Papentin complexity

In the present study the one-dimensional Papentin complexity measure was extended to two-dimensional patterns in the same manner as the Alexander-Carey extension. That is, the measure $L_1(n)$ defined in Equation 1 was calculated for each row, column, positive-slope diagonal and negative-slope diagonal of the pixels of a pattern. The resulting individual values may then be added to obtain measures for only vertical and horizontal complexities, only diagonal complexities, or the complexity of all four directions together.

4 Experimental Results

4.1 The Chipman dataset

The Chipman dataset [4] consists of 45 patterns, each composed of 12 black equal-size squares on a white background. The black squares are obtained by selecting 12 squares from a square 6×6 matrix or grid. Three patterns from this dataset are illustrated in Figure 2. The top row shows the patterns embedded in the 6×6 grid, along with their label at the top, and the complexity value obtained from the human judgments underneath. The bottom row shows how they were actually presented to the human subjects, without the underlying grid lines. Fifteen of the patterns were constructed so as to have simple structure, fifteen have complex structures, and fifteen were constructed by selecting 12 out of the 36 squares at random. The rightmost pattern is one such random pattern. The subjects consisted of eight graduate students from the Psychology Department at Harvard University. The reader is referred to the paper by Chipman for further details of the experimental procedure. Although eight is minimal number of subjects, it was large enough to obtain statistically significant results.

Table 1. Spearman rank correlation coefficients.

Symmetry Type	SS-W	SS	PL_1
Vertical-M	$-0{\cdot}843^{**}$	$-0{\cdot}831^{**}$	$0{\cdot}340^{**}$
Horizontal-M	$-0{\cdot}804^{**}$	$-0{\cdot}807^{**}$	$0{\cdot}630^{**}$
Diagonal-PS-M	$-0{\cdot}097$	$-0{\cdot}086$	$-0{\cdot}109$
Diagonal-NS-M	$-0{\cdot}002$	$0{\cdot}002$	$-0{\cdot}035$
90°	$-0{\cdot}531^{**}$	$-0{\cdot}532^{**}$	$0{\cdot}630^{**}$
180°	$-0{\cdot}606^{**}$	$-0{\cdot}519^{**}$	$0{\cdot}340^{**}$
270°	$-0{\cdot}531^{**}$	$-0{\cdot}532^{**}$	$0{\cdot}630^{**}$
Ver+Hor	$-0{\cdot}865^{**}$	$-0{\cdot}872^{**}$	$0{\cdot}725^{**}$
DPS+DNS	$-0{\cdot}026$	$0{\cdot}006$	$-0{\cdot}075$
V+H+Diagonals	$-0{\cdot}730$	$-0{\cdot}727^{**}$	$0{\cdot}284^{*}$
90° + 180° + 270°	$-0{\cdot}653^{**}$	$-0{\cdot}583^{**}$	$0{\cdot}700^{**}$
TOTAL	$-0{\cdot}734^{**}$	$-0{\cdot}689^{**}$	$0{\cdot}601^{**}$

$p < 0.05$ and $p < 0.01$ are indicated by * and **

4.2 Calculation of the complexity measures

Three measures of complexity were calculated for each of the 45 patterns in the Chipman dataset, under seven different symmetry operations of the patterns. These operations consisted of (1) vertical mirror symmetry (Vertical-M), (2) horizontal mirror symmetry (Horizontal-M), (3) mirror symmetry about a diagonal line of positive slope (Diagonal-PS-M), (4) mirror symmetry about a diagonal line of negative slope (Diagonal-NS-M), (5) rotation by 90 degrees (90°), (6)

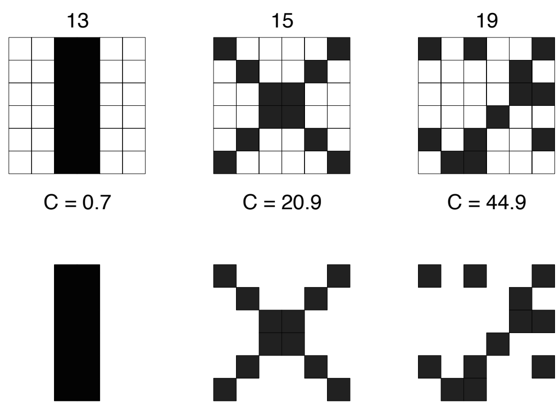

Figure 2. Three patterns from the Chipman data.

rotation by 180 degrees (180°), and (7) rotation by 270 degrees (270°). These calculations produced rankings of the 45 patterns by each complexity measure. These rankings were then compared with each other and with the ranking obtained from the human judgments by means of Spearman rank correlation coefficients between the rankings. The results are shown in Table 1, where SS-W denotes the weighted sub-symmetry measure used by Chipman [4], SS denotes the original sub-symmetry measure of Alexander and Carey [2], and PL_1 denotes the $L_1(n)$ complexity measure of Papentin [12], [13] calculated using Equation 1. The table also lists five combined scores for: the vertical plus horizontal sub-symmetries, the sum of the diagonal scores, the sum of the vertical, horizontal and diagonal scores, and finally, the sum of the four mirror symmetries and the three rotational symmetries (TOTAL). The statistical significance levels of $p < 0.05$ and $p < 0.01$ are indicated, respectively, by * and **. Regarding the computation of the Papentin $L_1(n)$ complexity, the description lengths were calculated such that the one-dimensional sequences examined were orthogonal to the axes of mirror symmetries. For example, for the Vertical Mirror symmetry (vertical axis), the sub-symmetries were calculated horizontally along the rows of the pattern, and so were the Papentin complexities, making it more convenient to compare compression with symmetry.

5 Conclusions and Further Research

Several conclusions are clearly evident from Table 1. For all three complexity measures, and the seven individual symmetry transformations, the correlation with human judgements is highest for vertical mirror symmetry, then for horizontal mirror symmetry, and finally for rotational symmetries. These results support psychological evidence that the most salient reflection symmetries perceived by the brain are: foremost, reflection about the vertical axis [19], followed a horizontal axis, and then diagonal axes [11]. The highest correlations were obtained for the sum of vertical and horizontal mirror symmetries. The weighted sub-symmetry measure of Chipman performed slightly better than the Alexander-Carey measure, suggesting that placing more weight on the larger sub-symmetries is a good strategy. Although the Papentin $L_1(n)$ complexity is highly and significantly correlated with human judgments of complexity, both sub-symmetry measures outperform the Papentin $L_1(n)$ complexity in this regard. The patterns used in this study were small (6×6), and a natural question arises as to how the results presented here will scale up for larger images. It is possible that for larger patterns, complexity measures based on higher hierarchical levels in Papentin's framework may yield shorter descriptions than $L_1(n)$, and may become competitive with sub-symetries in terms of modeling human judgments of visual complexity. Such studies are planned for the future.

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